A SECOND ORDER ESTIMATE FOR GENERAL COMPLEX HESSIAN EQUATIONS [1]

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Abstract

We derive a priori C^2 estimates for the χ-plurisubharmonic solutions of general complex Hessian equations with right-hand side depending on gradients.

1 Introduction

Let (X, ω) be a compact Kähler manifold of dimension $n \geq 2$. Let $u \in C^\infty(X)$ and consider a $(1,1)$ form $\chi(z,u)$ possibly depending on u and satisfying the positivity condition $\chi \geq \varepsilon\omega$ for some $\varepsilon > 0$. We define

$$g = \chi(z,u) + i\partial\bar\partial u, \tag{1.1}$$

and u is called χ-plurisubharmonic if $g > 0$ as a $(1,1)$ form. In this paper, we are concerned with the following complex Hessian equation, for $1 \leq k \leq n$,

$$\Big(\chi(z,u) + i\partial\bar\partial u\Big)^k \wedge \omega^{n-k} = \psi(z, Du, u)\, \omega^n, \tag{1.2}$$

where $\psi(z,v,u) \in C^\infty\left((T^{1,0}(X))^* \times \mathbf{R}\right)$ is a given strictly positive function.

The complex Hessian equation can be viewed as an intermediate equation between the Laplace equation and the complex Monge-Ampère equation. It encompasses the most natural invariants of the complex Hessian matrix of a real valued function, namely the elementary symmetric polynomials of its eigenvalues. When $k = 1$, equation (1.2) is quasilinear, and the estimates follow from the classical theory of quasilinear PDE. The real counterparts of (1.2) for $1 < k \leq n$, with ψ not depending on the gradient of u, have been studied extensively in the literature (see the survey paper [23] and more recent related work [7]), as these equations appear naturally and play very important roles in both classical and conformal geometry. When the right-hand side ψ depends on the gradient of the solution, even the real case has been a long standing problem due to substantial difficulties in obtaining a priori C^2 estimates. This problem was recently solved by Guan-Ren-Wang [10] for convex solutions of real Hessian equations.

In the complex case, the equation (1.2) with $\psi = \psi(z,u)$ has been extensively studied in recent years, due to its appearance in many geometric problems, including the J-flow [18] and quaternionic geometry [1]. The related Dirichlet problem for equation (1.2) on domains in $\mathbf{C}^n$ has been studied by Li [15] and Blocki [3]. The corresponding problem

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on compact Kähler or Hermitian manifolds has also been studied extensively, see, for example, [4, 11, 13, 16, 25]. In particular, as a crucial step in the continuity method, C^2 estimates for complex Hessian type equations have been studied in various settings, see [12, 20, 21, 22, 24].

However, the equation (1.2) with $\psi = \psi(z, Du, u)$ has been much less studied. An important case corresponding to $k = n = 2$, so that it is actually a Monge-Ampère equation in two dimensions, is central to the solution by Fu and Yau [5, 6] of a Strominger system on a toric fibration over a K3 surface. A natural generalization of this case to general dimension n was suggested by Fu and Yau [5] and can be expressed as

$$\Big(\Big(e^u + fe^{-u}\Big)\,\omega + n\, i\partial\bar\partial u\Big)^2 \wedge \omega^{n-2} = \psi(z, Du, u)\,\omega^n, \tag{1.3}$$

where $\psi(z, v, u)$ is a function on $(T^{1,0}(X))^* \times \mathbf{R}$ with a particular structure, and (X, ω) is a compact Kähler manifold. A priori estimates for this equation were obtained by the authors in [17].

In this paper, motivated by our previous work [17], we study a priori C^2 estimate for the equation (1.2) with general $\chi(z, u)$ and general right hand side $\psi(z, Du, u)$. Building on the techniques developed by Guan-Ren-Wang in [10] (see also [14]) for real Hessian equations), we can prove the following theorem.

Theorem 1 *Let (X, ω) be a compact Kähler manifold of complex dimension n. Suppose $u \in C^4(X)$ is a solution of equation (1.2) with $g = \chi + i\partial\bar\partial u > 0$ and $\chi(z, u) \geq \varepsilon\omega$. Let $0 < \psi(z, v, u) \in C^\infty((T^{1,0}X)^* \times \mathbf{R})$. Then we have the following uniform second order derivative estimate*

$$|D\bar D u|_\omega \leq C, \tag{1.4}$$

where C is a positive constant depending only on ε, n, k, $\sup_X |u|$, $\sup_X |Du|$, and the C^2 norm of χ as a function of (u, z), the infimum of ψ, and the C^2 norm of ψ as a function of (z, Du, u), all restricted to the ranges in Du and u defined by the uniform upper bounds on $|u|$ and $|Du|$.

We remark that the above estimate is stated for χ-plurisubharmonic solutions, that is, $g = \chi + i\partial\bar\partial u > 0$. Actually, we only need to assume that $g \in \Gamma_{k+1}$ cone (see (3.11) below for the definition of the Garding cone Γ_k and also the discussion in Remark 1 at the end of the paper). However, a better condition would be $g \in \Gamma_k$, which is the natural cone for ellipticity. In fact, this is still an open problem even for real Hessian equations when $2 < k < n$. If $k = 2$, Guan-Ren-Wang [10] removed the convexity assumption by investigating the structure of the operator. A simpler argument was given recently by Spruck-Xiao [19]. However, the arguments are not applicable to the complex case due to the difference between the terms $|DDu|^2$ and $|D\bar D u|^2$ in the complex setting. When $k = 2$

in the complex setting, C^2 estimates for equation (1.3) were obtained in [17] without the plurisubharmonicity assumption, but the techniques rely on the specific right hand side $\psi(z, Du, u)$ studied there.

We also note that if $k = n$, the condition $g = \chi + i\partial\bar{\partial}u > 0$ is the natural assumption for the ellipticity of equation (1.2). Thus, our result implies the a priori C^2 estimate for complex Monge-Ampère equations with right hand side depending on gradients:

$$\Big(\chi(z,u) + i\partial\bar{\partial}u\Big)^n = \psi(z, Du, u)\,\omega^n.$$

This generalizes the C^2 estimate for the equation studied by Fu and Yau [5, 6] mentioned above, which corresponds to $n = 2$ and a specific form $\chi(z, u)$ as well as a specific right hand side $\psi(z, Du, u)$. For dimension $n \geq 2$ and $k = n$, the estimate was obtained by Guan-Ma [8] using a different method where the structure of the Monge-Ampère operator plays an important role.

Compared to the estimates when $\psi = \psi(z, u)$, the dependence on the gradient of u in the equation (1.2) creates substantial new difficulties. The main obstacle is the appearance of terms such as $|DDu|^2$ and $|D\bar{D}u|^2$ when one differentiates the equation twice. We adapt the techniques used in [10] and [14] for real Hessian equations to overcome these difficulties. Furthermore, we also need to handle properly some subtle issues when dealing with the third order terms due to complex conjugacy.

2 Preliminaries

Let σ_k be the k-th elementary symmetric function, that is, for $1 \leq k \leq n$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{R}^n$,

$$\sigma_k(\lambda) = \sum_{1<i_1<\cdots<i_k<n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}.$$

Let $\lambda(a_{\bar{j}i})$ denote the eigenvalues of a Hermitian symmetric matrix $(a_{\bar{j}i})$ with respect to the background Kähler metric ω. We define $\sigma_k(a_{\bar{j}i}) = \sigma_k\Big(\lambda(a_{\bar{j}i})\Big)$. This definition can be naturally extended to complex manifolds. Denoting $A^{1,1}(X)$ to be the space of smooth real $(1,1)$-forms on a compact Kähler manifold (X, ω), we define for any $g \in A^{1,1}(X)$,

$$\sigma_k(g) = \binom{n}{k}\frac{g^k \wedge \omega^{n-k}}{\omega^n}.$$

Using the above notation, we can re-write equation (1.2) as following:

$$\sigma_k(g) = \sigma_k(\chi_{\bar{j}i} + u_{\bar{j}i}) = \psi(z, Du, u). \tag{2.1}$$

We will use the notation

$$\sigma_k^{p\bar q} = \frac{\partial \sigma_k(g)}{\partial g_{\bar q p}}, \qquad \sigma_k^{p\bar q, r\bar s} = \frac{\partial^2 \sigma_k(g)}{\partial g_{\bar q p} \bar\partial g_{\bar s r}}.$$

The symbol D will indicate the covariant derivative with respect to the given metric ω. All norms and inner products will be with respect to ω unless denoted otherwise. We will denote by $\lambda_1, \lambda_2, \cdots, \lambda_n$ the eigenvalues of $g_{\bar j i} = \chi_{\bar j i} + u_{\bar j i}$ with respect to ω, and use the ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Our calculations will be carried out at a point z on the manifold X, and we shall use coordinates such that at this point $\omega = i \sum \delta_{\ell k}\, dz^k \wedge d\bar z^\ell$ and $g_{\bar j i}$ is diagonal. We will also use the notation

$$\mathcal{F} = \sum_p \sigma_k^{p\bar p}. \tag{2.2}$$

Differentiating equation (2.1) yields

$$\sigma_k^{p\bar q} D_{\bar j} g_{\bar q p} \;\; = \;\; D_{\bar j}\psi. \tag{2.3}$$

Differentiating the equation a second time gives

$$\begin{aligned} &\sigma_k^{p\bar q} D_i D_{\bar j} g_{\bar q p} + \sigma_k^{p\bar q, r\bar s} D_i g_{\bar q p} D_{\bar j} g_{\bar s r} = D_i D_{\bar j}\psi \\ \geq \;\; & -C(1 + |DDu|^2 + |D\bar D u|^2) + \sum_\ell \psi_{v_\ell} u_{\ell \bar j i} + \sum_\ell \psi_{\bar v_\ell} u_{\bar\ell \bar j i}. \end{aligned} \tag{2.4}$$

We will denote by C a uniform constant which depends only on (X, ω), n, k, $\|\chi\|_{C^2}$, $\inf\psi$, $\|u\|_{C^1}$ and $\|\psi\|_{C^2}$. We now compute the operator $\sigma_k^{p\bar q} D_p D_{\bar q}$ acting on $g_{\bar j i} = \chi_{\bar j i} + u_{\bar j i}$. Recalling that $\chi_{\bar j i}$ depends on u, we estimate

$$\begin{aligned} \sigma_k^{p\bar q} D_p D_{\bar q} g_{\bar j i} &= \sigma_k^{p\bar q} D_p D_{\bar q} D_i D_{\bar j} u + \sigma_k^{p\bar q} D_p D_{\bar q} \chi_{\bar j i} \\ &\geq \sigma_k^{p\bar q} D_p D_{\bar q} D_i D_{\bar j} u - C(1+\lambda_1)\mathcal{F}. \end{aligned} \tag{2.5}$$

Commuting derivatives

$$\begin{aligned} D_p D_{\bar q} D_i D_{\bar j} u &= D_i D_{\bar j} D_p D_{\bar q} u - R_{\bar q i \bar j}{}^{\bar a} u_{\bar a p} + R_{\bar q p \bar j}{}^{\bar a} u_{\bar a i} \\ &= D_i D_{\bar j} g_{\bar q p} - D_i D_{\bar j} \chi_{\bar q p} - R_{\bar q i \bar j}{}^{\bar a} u_{\bar a p} + R_{\bar q p \bar j}{}^{\bar a} u_{\bar a i}. \end{aligned} \tag{2.6}$$

Therefore, by (2.4),

$$\begin{aligned} \sigma_k^{p\bar q} D_p D_{\bar q} g_{\bar j i} \;\; \geq \;\; & -\sigma_k^{p\bar q, r\bar s} D_j g_{\bar q p} D_{\bar j} g_{\bar s r} + \sum \psi_{v_\ell} g_{\bar j i \ell} + \sum \psi_{\bar v_\ell} g_{\bar j i \bar\ell} \\ & -C(1 + |DDu|^2 + |D\bar D u|^2 + (1+\lambda_1)\mathcal{F}). \end{aligned} \tag{2.7}$$

We next compute the operator $\sigma_k^{p\bar q} D_p D_{\bar q}$ acting on $|Du|^2$. Introduce the notation

$$|DDu|^2_{\sigma\omega} = \sigma_k^{p\bar q} \omega^{m\bar\ell} D_p D_m u D_{\bar q} D_{\bar\ell} u, \;\; |D\bar D u|^2_{\sigma\omega} = \sigma_k^{p\bar q} \omega^{m\bar\ell} D_p D_{\bar\ell} u D_m D_{\bar q} u. \tag{2.8}$$

Then

$$
\begin{aligned}
\sigma_k^{p\bar{q}}|Du|^2_{\bar{q}p} &= \sigma_k^{p\bar{q}}(D_pD_{\bar{q}}D_muD^mu + D_muD_pD_{\bar{q}}D^mu) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega} \\
&= \sigma_k^{p\bar{q}}\{D_m(g_{\bar{q}p} - \chi_{\bar{q}p})D^mu + D_muD^m(g_{\bar{q}p} - \chi_{\bar{q}p})\} + \sigma_k^{p\bar{q}}R_{\bar{q}p}{}^{m\bar{\ell}}u_{\bar{\ell}}u_m \\
&\quad + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}. \qquad (2.9)
\end{aligned}
$$

Using the differentiated equation we obtain

$$
\begin{aligned}
\sigma_k^{p\bar{q}}|Du|^2_{\bar{q}p} &\geq 2\mathrm{Re}\langle Du, D\psi\rangle - C(1+\mathcal{F}) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega} \\
&\geq 2\mathrm{Re}\{\sum_{p,m}(D_pD_muD_{\bar{p}}u + D_puD_{\bar{p}}D_mu)\psi_{v_m}\} - C(1+\mathcal{F}) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}.
\end{aligned}
$$

To simplify the expression, we introduce the notation

$$
\langle D|Du|^2, D_{\bar{v}}\psi\rangle = \sum_m (D_mD_puD^pu\,\psi_{v_m} + D_puD_mD^pu\,\psi_{v_m}). \qquad (2.10)
$$

We obtain

$$
\sigma_k^{p\bar{q}}|Du|^2_{\bar{q}p} \geq 2\mathrm{Re}\langle D|Du|^2, D_{\bar{v}}\psi\rangle - C(1+\mathcal{F}) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}. \qquad (2.11)
$$

We also compute

$$
-\sigma_k^{p\bar{q}}u_{\bar{q}p} = \sigma_k^{p\bar{q}}(\chi_{\bar{q}p} - g_{\bar{q}p}) \geq \varepsilon\mathcal{F} - k\psi. \qquad (2.12)
$$

3 The C^2 estimate

In this section, we give the proof of the estimate stated in the theorem. When $k = 1$, the equation (1.2) becomes

$$
\Delta_\omega u + \mathrm{Tr}_\omega\chi(z,u) = n\psi(z, Du, u) \qquad (3.1)
$$

where Δ_ω and Tr_ω are the Laplacian and trace with respect to the background metric ω. It follows that $\Delta_\omega u$ is bounded, and the desired estimate follows in turn from the positivity of the metric g. Henceforth, we assume that $k \geq 2$. Motivated by the idea from [10] for real Hessian equations, we apply the maximum principle to the following test function:

$$
G = \log P_m + mN|Du|^2 - mMu, \qquad (3.2)
$$

where $P_m = \sum_j \lambda_j^m$. Here, m, M and N are large positive constants to be determined later. We may assume that the maximum of G is achieved at some point $z \in X$. After rotating the coordinates, we may assume that the matrix $g_{\bar{j}i} = \chi_{\bar{j}i} + u_{\bar{j}i}$ is diagonal.

Recall that if $F(A) = f(\lambda_1, \cdots, \lambda_n)$ is a symmetric function of the eigenvalues of a Hermitian matrix $A = (a_{\bar{j}i})$, then at a diagonal matrix A with distinct eigenvalues, we have (see [2]),

$$F^{i\bar{j}} = \delta_{ij} f_i, \tag{3.3}$$

$$F^{i\bar{j},r\bar{s}} w_{i\bar{j}k} w_{r\bar{s}\bar{k}} = \sum f_{ij} w_{i\bar{i}k} w_{j\bar{j}\bar{k}} + \sum_{p\neq q} \frac{f_p - f_q}{\lambda_p - \lambda_q} |w_{p\bar{q}k}|^2. \tag{3.4}$$

where $F^{i\bar{j}} = \frac{\partial F}{\partial a_{\bar{j}i}}$, $F^{i\bar{j},r\bar{s}} = \frac{\partial^2 F}{\partial a_{\bar{j}i} \partial a_{\bar{s}r}}$, and $w_{i\bar{j}k}$ is an arbitrary tensor. Using these identities to differentiate G, we first obtain the critical equation

$$\frac{DP_m}{P_m} + mND|Du|^2 - mMDu = 0. \tag{3.5}$$

Differentiating G a second time and contracting with $\sigma_k^{p\bar{q}}$ yields

$$\begin{aligned} 0 \ \geq \ & \frac{m}{P_m}\Big\{ \sum_j \lambda_j^{m-1} \sigma_k^{p\bar{p}} D_p D_{\bar{p}} g_{\bar{j}j} \Big\} - \frac{|DP_m|_\sigma^2}{P_m^2} + mN\sigma_k^{p\bar{p}} |Du|^2_{\bar{p}p} - mM\sigma_k^{p\bar{p}} u_{\bar{p}p} \\ & + \frac{m}{P_m}\Big\{ (m-1)\sum_j \lambda_j^{m-2} \sigma_k^{p\bar{p}} |D_p g_{\bar{j}j}|^2 + \sigma_k^{p\bar{p}} \sum_{i\neq j} \frac{\lambda_i^{m-1} - \lambda_j^{m-1}}{\lambda_i - \lambda_j} |D_p g_{\bar{j}i}|^2 \Big\}. \end{aligned} \tag{3.6}$$

Here, we used the notation $|\eta|_\sigma^2 = \sigma_k^{p\bar{q}} \eta_p \eta_{\bar{q}}$. Substituting (2.7), (2.11) and (2.12)

$$\begin{aligned} 0 \ \geq \ & \frac{1}{P_m}\Big\{ -C\sum_j \lambda_j^{m-1} (1 + |DDu|^2 + |D\bar{D}u|^2 + (1+\lambda_1)\mathcal{F}) \Big\} \\ & + \frac{1}{P_m}\Big\{ \sum_j \lambda_j^{m-1} (-\sigma_k^{p\bar{q},r\bar{s}} D_j g_{\bar{q}p} D_{\bar{j}} g_{\bar{s}r} + \sum_\ell \psi_{v_\ell} g_{\bar{j}j\ell} + \sum_\ell \psi_{\bar{v}_\ell} g_{\bar{j}j\bar{\ell}}) \Big\} \\ & + \frac{1}{P_m}\Big\{ (m-1)\sum_j \lambda_j^{m-2} \sigma_k^{p\bar{p}} |D_p g_{\bar{j}j}|^2 + \sigma_k^{p\bar{p}} \sum_{i\neq j} \frac{\lambda_i^{m-1} - \lambda_j^{m-1}}{\lambda_i - \lambda_j} |D_p g_{\bar{j}i}|^2 \Big\} \\ & - \frac{|DP_m|_\sigma^2}{mP_m^2} + N(|DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}) \\ & + N\langle D|Du|^2, D_{\bar{v}}\psi\rangle + N\langle D_{\bar{v}}\psi, D|Du|^2\rangle + (M\varepsilon - CN)\mathcal{F} - kM\psi. \end{aligned} \tag{3.7}$$

From the critical equation (3.5), we have

$$\frac{1}{P_m} \sum_{j,\ell} \lambda_j^{m-1} g_{\bar{j}j\ell} \psi_{v_\ell} \ = \ \frac{1}{m} \langle \frac{DP_m}{P_m}, D_{\bar{v}}\psi \rangle = -N\langle D|Du|^2, D_{\bar{v}}\psi\rangle + M\langle Du, D_{\bar{v}}\psi\rangle.$$

It follows that

$$\begin{aligned} & \frac{1}{P_m} \sum_{j,\ell} \Big(\psi_{v_\ell} g_{\bar{j}j\ell} + \psi_{\bar{v}_\ell} g_{\bar{j}j\bar{\ell}} \Big) + N\langle D|Du|^2, D_{\bar{v}}\psi\rangle + N\langle D_{\bar{v}}\psi, D|Du|^2\rangle \\ = \ & M\left(\langle Du, D_{\bar{v}}\psi\rangle + \langle D_{\bar{v}}\psi, Du\rangle\right) \geq -CM. \end{aligned}$$

Using (3.4), one can obtain the well-known identity

$$-\sigma_k^{p\bar{q},r\bar{s}} D_j g_{\bar{q}p} D_{\bar{j}} g_{\bar{s}r} = -\sigma_k^{p\bar{p},q\bar{q}} D_j g_{\bar{p}p} D_{\bar{j}} g_{\bar{q}q} + \sigma_k^{p\bar{p},q\bar{q}} |D_j g_{\bar{p}q}|^2, \tag{3.8}$$

where $\sigma_k^{p\bar{p},q\bar{q}} = \frac{\partial}{\partial\lambda_p}\frac{\partial}{\partial\lambda_q}\sigma_k(\lambda)$. We assume that $\lambda_1 \gg 1$, otherwise the C^2 estimate is complete. The main inequality (3.7) becomes

$$\begin{aligned} 0 \ \geq\ & \frac{-C}{\lambda_1}\Big\{1 + |DDu|^2 + |D\bar{D}u|^2\Big\} + \frac{1}{P_m}\Big\{\sum_j \lambda_j^{m-1}(-\sigma_k^{p\bar{p},q\bar{q}} D_j g_{\bar{p}p} D_{\bar{j}} g_{\bar{q}q} + \sigma_k^{p\bar{p},q\bar{q}} |D_j g_{\bar{p}q}|^2\Big\} \\ & + \frac{1}{P_m}\Big\{(m-1)\sum_j \lambda_j^{m-2}\sigma_k^{p\bar{p}}|D_p g_{\bar{j}j}|^2 + \sigma_k^{p\bar{p}}\sum_{i\neq j}\frac{\lambda_i^{m-1} - \lambda_j^{m-1}}{\lambda_i - \lambda_j}|D_p g_{\bar{j}i}|^2\Big\} \\ & - \frac{|DP_m|_\sigma^2}{mP_m^2} + N(|DDu|_{\sigma\omega}^2 + |D\bar{D}u|_{\sigma\omega}^2) + (M\varepsilon - CN - C)\mathcal{F} - CM. \end{aligned} \tag{3.9}$$

The main objective is to show that the third order terms on the right hand side of (3.9) are nonnegative. To deal with this issue, we need a lemma from [10] (see also [9, 14]).

Lemma 1 ([10]) *Suppose $1 \leq \ell < k \leq n$, and let $\alpha = 1/(k-\ell)$. Let $W = (w_{\bar{q}p})$ be a Hermitian tensor in the Γ_k cone. Then, for any $\theta > 0$,*

$$\begin{aligned} & -\sigma_k^{p\bar{p},q\bar{q}}(W) w_{\bar{p}pi} w_{\bar{q}q\bar{i}} + (1 - \alpha + \frac{\alpha}{\theta})\frac{|D_i\sigma_k(W)|^2}{\sigma_k(W)} \\ \geq\ & \sigma_k(W)(\alpha + 1 - \alpha\theta)\Big|\frac{D_i\sigma_\ell(W)}{\sigma_\ell(W)}\Big|^2 - \frac{\sigma_k}{\sigma_\ell}(W)\sigma_\ell^{p\bar{p},q\bar{q}}(W) w_{\bar{p}pi} w_{\bar{q}q\bar{i}}. \end{aligned} \tag{3.10}$$

Here the Γ_k cone is defined as following:

$$\Gamma_k = \ \{\lambda \in \mathbf{R}^n \mid \sigma_m(\lambda) > 0,\ m = 1, \cdots, k\}. \tag{3.11}$$

We say a Hermitian matrix $W \in \Gamma_k$ if $\lambda(W) \in \Gamma_k$.

It follows from the above lemma that, by taking $\ell = 1$, we have

$$-\sigma_k^{p\bar{p},q\bar{q}} D_i g_{\bar{p}p} D_{\bar{i}} g_{\bar{q}q} + K|D_i\sigma_k|^2 \geq 0, \tag{3.12}$$

for $K > (1 - \alpha + \frac{\alpha}{\theta})\,(\inf\psi)^{-1}$ if $2 \leq k \leq n$.

We shall denote

$$A_i = \frac{\lambda_i^{m-1}}{P_m}\Big\{K|D_i\sigma_k|^2 - \sigma_k^{p\bar{p},q\bar{q}} D_i g_{\bar{p}p} D_{\bar{i}} g_{\bar{q}q}\Big\},$$

$$B_i = \frac{1}{P_m}\Big\{\sum_p \sigma_k^{p\bar{p},i\bar{i}}\lambda_p^{m-1}|D_i g_{\bar{p}p}|^2\Big\}, \quad C_i = \frac{(m-1)\sigma_k^{i\bar{i}}}{P_m}\Big\{\sum_p \lambda_p^{m-2}|D_i g_{\bar{p}p}|^2\Big\},$$

$$D_i=\frac{1}{P_m}\Big\{\sum_{p\neq i}\sigma_k^{p\bar p}\,\frac{\lambda_p^{m-1}-\lambda_i^{m-1}}{\lambda_p-\lambda_i}|D_ig_{\bar pp}|^2\Big\},\qquad E_i=\frac{m\sigma_k^{i\bar i}}{P_m^2}\Big|\sum_p\lambda_p^{m-1}D_ig_{\bar pp}\Big|^2.$$

Define $T_{j\bar pq}=D_j\chi_{\bar pq}-D_q\chi_{\bar pj}$. For any $0<\tau<1$, we can estimate

$$\begin{aligned}\frac{1}{P_m}\Big\{\sum_p\lambda_p^{m-1}\sigma_k^{j\bar j,i\bar i}|D_pg_{\bar ji}|^2\Big\} &\geq \frac{1}{P_m}\Big\{\sum_p\lambda_p^{m-1}\sigma_k^{p\bar p,i\bar i}|D_ig_{\bar pp}+T_{p\bar pi}|^2\Big\}\\ &\geq \frac{1}{P_m}\Big\{\sum_p\lambda_p^{m-1}\sigma_k^{p\bar p,i\bar i}\{(1-\tau)|D_ig_{\bar pp}|^2-C_\tau|T_{p\bar pi}|^2\}\Big\}\\ &= (1-\tau)\sum_i B_i-\frac{C_\tau}{P_m}\sum_p\lambda_p^{m-2}(\lambda_p\sigma_k^{p\bar p,i\bar i})|T_{p\bar pi}|^2.\end{aligned}$$

Now, we use $\sigma_l(\lambda|i)$ and $\sigma_l(\lambda|ij)$ to denote the l-th elementary function of

$$(\lambda|i)=(\lambda_1,\cdots,\widehat{\lambda_i},\cdots,\lambda_n)\in\mathbf{R}^{n-1}\ \ \textit{and}\ \ (\lambda|ij)=(\lambda_1,\cdots,\widehat{\lambda_i},\cdots,\widehat{\lambda_j},\cdots,\lambda_n)\in\mathbf{R}^{n-2}$$

respectively. The following simple identities are used frequently,

$$\sigma_k^{i\bar i}=\sigma_{k-1}(\lambda|i),\qquad \sigma_k^{p\bar p,i\bar i}=\sigma_{k-2}(\lambda|pi).$$

Using the identity $\sigma_l(\lambda)=\sigma_l(\lambda|p)+\lambda_p\sigma_{l-1}(\lambda|p)$ for any $1\leq p\leq n$, we obtain

$$\begin{aligned}\frac{1}{P_m}\Big\{\sum_p\lambda_p^{m-1}\sigma_k^{j\bar j,i\bar i}|D_pg_{\bar ji}|^2\Big\} &\geq (1-\tau)\sum_i B_i-\frac{C_\tau}{P_m}\sum_p\lambda_p^{m-2}(\sigma_k^{i\bar i}-\sigma_{k-1}(\lambda|pi))|T_{p\bar pi}|^2\\ &\geq (1-\tau)\sum_i B_i-\frac{C_\tau}{\lambda_1^2}\mathcal{F}\geq(1-\tau)\sum_i B_i-\mathcal{F}. \qquad (3.13)\end{aligned}$$

We used the notation C_τ for a constant depending on τ. To get the last inequality above, we assumed that $\lambda_1^2\geq C_\tau$; otherwise, we already have the desired estimate $\lambda_1\leq C$. Similarly, we may estimate

$$\begin{aligned}\frac{1}{P_m}\sigma_k^{j\bar j}\sum_{i\neq p}\frac{\lambda_i^{m-1}-\lambda_p^{m-1}}{\lambda_i-\lambda_p}|D_jg_{\bar pi}|^2 &\geq \frac{1}{P_m}\sigma_k^{p\bar p}\sum_{i;p\neq i}\frac{\lambda_i^{m-1}-\lambda_p^{m-1}}{\lambda_i-\lambda_p}|D_ig_{\bar pp}+T_{p\bar pi}|^2 \qquad (3.14)\\ &\geq \frac{1}{P_m}\sigma_k^{p\bar p}\sum_{i;p\neq i}\frac{\lambda_i^{m-1}-\lambda_p^{m-1}}{\lambda_i-\lambda_p}\{(1-\tau)|D_ig_{\bar pp}|^2-C_\tau|T_{p\bar pi}|^2\}\\ &\geq \sum_i(1-\tau)D_i-\frac{C_\tau}{\lambda_1^2}\mathcal{F}\geq\sum_i(1-\tau)D_i-\mathcal{F}.\end{aligned}$$

With the introduced notation in place, the main inequality becomes

$$\begin{aligned}0\ \geq\ &\frac{-C(K)}{\lambda_1}\Big\{1+|DDu|^2+|D\bar Du|^2\Big\}-\tau\frac{|DP_m|_\sigma^2}{mP_m^2}\\ &+\sum_i\Big\{A_i+(1-\tau)B_i+C_i+(1-\tau)D_i-(1-\tau)E_i\Big\}\\ &+N(|DDu|_{\sigma\omega}^2+|D\bar Du|_{\sigma\omega}^2)+(M\varepsilon-CN-C)\mathcal{F}-CM. \qquad (3.15)\end{aligned}$$

Using the critical equation (3.5), we have

$$\begin{aligned}\tau\frac{|DP_m|^2_\sigma}{mP_m^2} &= \tau m\Big|ND|Du|^2 - MDu\Big|^2_\sigma \le 2\tau m(N^2|D|Du|^2|^2_\sigma + M^2|Du|^2_\sigma)\\ &\le C\tau mN^2(|DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}) + C\tau mM^2\mathcal{F}. \qquad (3.16)\end{aligned}$$

We thus have

$$\begin{aligned}0 \ \ge\ & \frac{-C(K)}{\lambda_1}\Big\{1 + |DDu|^2 + |D\bar{D}u|^2\Big\} + (N - C\tau mN^2)(|DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega})\\ &+ \sum_i\Big\{A_i + (1-\tau)B_i + C_i + (1-\tau)D_i - (1-\tau)E_i\Big\}\\ &+ (M\varepsilon - C\tau mM^2 - CN - C)\mathcal{F} - CM. \qquad (3.17)\end{aligned}$$

3.1 Estimating the Third Order Terms

In this subsection, we will adapt the argument in [14] to estimate the third order terms.

Lemma 2 *For sufficiently large m, the following estimates hold:*

$$P_m^2(B_1 + C_1 + D_1 - E_1) \ge P_m\lambda_1^{m-2}\sum_{p\ne 1}\sigma_k^{p\bar{p}}|D_1 g_{\bar{p}p}|^2 - \lambda_1^m\sigma_k^{1\bar{1}}\lambda_1^{m-2}|D_1 g_{\bar{1}1}|^2, \qquad (3.18)$$

and for any fixed $i \ne 1$,

$$P_m^2(B_i + C_i + D_i - E_i) \ge 0. \qquad (3.19)$$

Proof. Fix $i \in \{1, 2, \dots, n\}$. First, we compute

$$\begin{aligned}P_m(B_i + D_i) &= \sum_{p\ne i}\sigma_k^{p\bar{p},i\bar{i}}\lambda_p^{m-1}|D_i g_{\bar{p}p}|^2 + \sum_{p\ne i}\sigma_k^{p\bar{p}}\ \frac{\lambda_p^{m-1} - \lambda_i^{m-1}}{\lambda_p - \lambda_i}|D_i g_{\bar{p}p}|^2\\ &= \sum_{p\ne i}\lambda_p^{m-2}\Big\{(\lambda_p\sigma_k^{p\bar{p},i\bar{i}} + \sigma_k^{p\bar{p}})|D_i g_{\bar{p}p}|^2\Big\} + \Big\{\sum_{p\ne i}\sigma_k^{p\bar{p}}\sum_{q=0}^{m-3}\lambda_p{}^q\lambda_i^{m-2-q}|D_i g_{\bar{p}p}|^2\Big\}.\end{aligned}$$

Note that,

$$\lambda_p\sigma_k^{p\bar{p},i\bar{i}} + \sigma_k^{p\bar{p}} \ge \sigma_k^{i\bar{i}}.$$

To see this, we write

$$\begin{aligned}\lambda_p\sigma_k^{p\bar{p},i\bar{i}} + \sigma_k^{p\bar{p}} &= \lambda_p\sigma_{k-2}(\lambda|pi) + \sigma_{k-1}(\lambda|p)\\ &= \sigma_{k-1}(\lambda|i) - \sigma_{k-1}(\lambda|ip) + \sigma_{k-1}(\lambda|p)\\ &= \sigma_{k-1}(\lambda|i) + \lambda_i\sigma_{k-2}(\lambda|ip) \ge \sigma_{k-1}(\lambda|i) = \sigma_k^{i\bar{i}},\end{aligned}$$

where we used the standard identity $\sigma_l(\lambda) = \sigma_l(\lambda|p) + \lambda_p \sigma_{l-1}(\lambda|p)$ twice, to get the second and third equalities. Therefore

$$P_m(B_i + D_i) \geq \sigma_k^{i\bar{i}} \Big\{ \sum_{p \neq i} \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 \Big\} + \Big\{ \sum_{p \neq i} \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} \lambda_p{}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2 \Big\}. \tag{3.20}$$

It follows that

$$\begin{aligned} P_m(B_i + C_i + D_i) \;\; &\geq \;\; m\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 + (m-1)\sigma_k^{i\bar{i}} \lambda_i^{m-2} |D_i g_{\bar{i}i}|^2 \\ &\quad + \sum_{p \neq i} \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} \lambda_p{}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2. \end{aligned} \tag{3.21}$$

Expanding out the definition of E_i

$$P_m^2 E_i = m\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_p^{2m-2} |D_i g_{\bar{p}p}|^2 + m\sigma_k^{i\bar{i}} \lambda_i^{2m-2} |D_i g_{\bar{i}i}|^2 + m\sigma_k^{i\bar{i}} \sum_p \sum_{q \neq p} \lambda_p^{m-1} \lambda_q^{m-1} D_i g_{\bar{p}p} D_{\bar{i}} g_{\bar{q}q}. \tag{3.22}$$

Therefore

$$\begin{aligned} & P_m^2(B_i + C_i + D_i - E_i) \\ \geq \;\; & \Big\{ m\sigma_k^{i\bar{i}} \sum_{p \neq i} (P_m - \lambda_p^m) \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 - m\sigma_k^{i\bar{i}} \sum_{p \neq i} \sum_{q \neq p,i} \lambda_p^{m-1} \lambda_q^{m-1} D_i g_{\bar{p}p} D_{\bar{i}} g_{\bar{q}q} \Big\} \\ & + P_m \sum_{p \neq i} \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} \lambda_p{}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2 - 2m\sigma_k^{i\bar{i}} \mathrm{Re} \sum_{q \neq i} \lambda_i^{m-1} \lambda_q^{m-1} D_i g_{\bar{i}i} D_{\bar{i}} g_{\bar{q}q} \\ & + \{(m-1)P_m - m\lambda_i^m\} \sigma_k^{i\bar{i}} \lambda_i^{m-2} |D_i g_{\bar{i}i}|^2. \end{aligned} \tag{3.23}$$

We shall estimate the expression in brackets. First,

$$m\sigma_k^{i\bar{i}} \sum_{p \neq i} (P_m - \lambda_p^m) \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 = m\sigma_k^{i\bar{i}} \sum_{p \neq i} \sum_{q \neq p,i} \lambda_q^m \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 + m\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_i^m \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2.$$

Next, we can estimate

$$\begin{aligned} & -m\sigma_k^{i\bar{i}} \sum_{p \neq i} \sum_{q \neq p,i} \lambda_p^{m-1} \lambda_q^{m-1} D_i g_{\bar{p}p} D_{\bar{i}} g_{\bar{q}q} \\ \geq \;\; & -m\sigma_k^{i\bar{i}} \sum_{p \neq i} \sum_{q \neq p,i} \frac{1}{2} \{ \lambda_p^{m-2} \lambda_q^m |D_i g_{\bar{p}p}|^2 + \lambda_p^m \lambda_q^{m-2} |D_i g_{\bar{q}q}|^2 \} = -m\sigma_k^{i\bar{i}} \sum_{p \neq i} \sum_{q \neq p,i} \lambda_p^{m-2} \lambda_q^m |D_i g_{\bar{p}p}|^2. \end{aligned} \tag{3.24}$$

We arrive at

$$\begin{aligned} & P_m^2(B_i + C_i + D_i - E_i) \\ \geq \;\; & m\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_i^m \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 + P_m \sum_{p \neq i} \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} \lambda_p{}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2 \\ & - 2m\sigma_k^{i\bar{i}} \mathrm{Re} \Big\{ \lambda_i^{m-1} D_i g_{\bar{i}i} \sum_{q \neq i} \lambda_q^{m-1} D_{\bar{i}} g_{\bar{q}q} \Big\} + \{(m-1)P_m - m\lambda_i^m\} \sigma_k^{i\bar{i}} \lambda_i^{m-2} |D_i g_{\bar{i}i}|^2. \end{aligned} \tag{3.25}$$

The next step is to extract good terms from the second summation on the first line. We fix a $p \neq i$.

Case 1: $\lambda_i \geq \lambda_p$. Then $\sigma_k^{p\bar{p}} \geq \sigma_k^{i\bar{i}}$. Hence

$$P_m \sigma_k^{p\bar{p}} \sum_{q=1}^{m-3} {\lambda_p}^q \lambda_i^{m-2-q} \geq \lambda_i^m \sigma_k^{i\bar{i}} \sum_{q=1}^{m-3} {\lambda_p}^q \lambda_p^{m-2-q} = (m-3)\sigma_k^{i\bar{i}} \lambda_i^m \lambda_p^{m-2}. \tag{3.26}$$

Case 2: $\lambda_i \leq \lambda_p$. Then $\lambda_p \sigma_k^{p\bar{p}} = \lambda_i \sigma_k^{i\bar{i}} + (\sigma_k(\lambda|i) - \sigma_k(\lambda|p)) \geq \lambda_i \sigma_k^{i\bar{i}}$, and we obtain

$$P_m \sigma_k^{p\bar{p}} \sum_{q=1}^{m-3} {\lambda_p}^q \lambda_i^{m-2-q} \geq \lambda_p^m \sigma_k^{i\bar{i}} \sum_{q=1}^{m-3} {\lambda_p}^{q-1} \lambda_i^{m-1-q} \geq (m-3)\sigma_k^{i\bar{i}} \lambda_i^m \lambda_p^{m-2}. \tag{3.27}$$

Combining both cases, we have

$$\begin{aligned} P_m \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} {\lambda_p}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2 &= P_m \sigma_k^{p\bar{p}} \sum_{q=1}^{m-3} {\lambda_p}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2 + P_m \sigma_k^{p\bar{p}} \lambda_i^{m-2} |D_i g_{\bar{p}p}|^2 \\ &\geq (m-3)\sigma_k^{i\bar{i}} \lambda_i^m \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 + P_m \sigma_k^{p\bar{p}} \lambda_i^{m-2} |D_i g_{\bar{p}p}|^2. \end{aligned}$$

Substituting this estimate into inequality (3.25), we obtain

$$\begin{aligned} & P_m^2 (B_i + C_i + D_i - E_i) \qquad (3.28) \\ \geq\ & (2m-3)\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_i^m \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 - 2m \sigma_k^{i\bar{i}} \mathrm{Re}\Big\{ \lambda_i^{m-1} D_i g_{\bar{i}i} \sum_{p \neq i} \lambda_p^{m-1} D_{\bar{i}} g_{\bar{p}p} \Big\} \\ & + P_m \lambda_i^{m-2} \sum_{p \neq i} \sigma_k^{p\bar{p}} |D_i g_{\bar{p}p}|^2 + \{(m-1)P_m - m\lambda_i^m\} \sigma_k^{i\bar{i}} \lambda_i^{m-2} |D_i g_{\bar{i}i}|^2. \end{aligned}$$

Choose $m \gg 1$ such that

$$m^2 \leq (2m-3)(m-2). \tag{3.29}$$

We can therefore estimate

$$\begin{aligned} & 2m \sigma_k^{i\bar{i}} \mathrm{Re}\Big\{ \lambda_i^{m-1} D_i g_{\bar{i}i} \sum_{p \neq i} \lambda_p^{m-1} D_{\bar{i}} g_{\bar{p}p} \Big\} \\ \leq\ & 2\sigma_k^{i\bar{i}} \sum_{p \neq i} \{(2m-3)^{1/2} \lambda_i^{m/2} \lambda_p^{\frac{m-2}{2}} |D_i g_{\bar{p}p}|\}\{(m-2)^{1/2} \lambda_i^{\frac{m-2}{2}} \lambda_p^{m/2} |D_{\bar{i}} g_{\bar{i}i}|\} \\ \leq\ & (2m-3)\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_i^m \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 + (m-2)\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_i^{m-2} \lambda_p^m |D_{\bar{i}} g_{\bar{i}i}|^2. \qquad (3.30) \end{aligned}$$

We finally arrive at

$$\begin{aligned} P_m^2 (B_i + C_i + D_i - E_i) \geq\ & P_m \lambda_i^{m-2} \sum_{p \neq i} \sigma_k^{p\bar{p}} |D_i g_{\bar{p}p}|^2 + \{(m-1)P_m - m\lambda_i^m\} \sigma_k^{i\bar{i}} \lambda_i^{m-2} |D_i g_{\bar{i}i}|^2 \\ & - (m-2)\sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_i^{m-2} \lambda_p^m |D_{\bar{i}} g_{\bar{i}i}|^2. \qquad (3.31) \end{aligned}$$

If we let $i=1$, we obtain inequality (3.18). For any fixed $i\neq 1$, this inequality yields

$$\begin{aligned}
P_m^2(B_i+C_i+D_i-E_i) &\geq P_m\lambda_i^{m-2}\sum_{p\neq i}\sigma_k^{p\bar p}|D_ig_{\bar pp}|^2+\{(m-1)\lambda_1^m-\lambda_i^m\}\sigma_k^{i\bar i}\lambda_i^{m-2}|D_ig_{\bar ii}|^2\\
&\quad +(m-1)\sum_{p\neq 1,i}\lambda_p^m\sigma_k^{i\bar i}\lambda_i^{m-2}|D_ig_{\bar ii}|^2-(m-2)\sigma_k^{i\bar i}\sum_{p\neq i}\lambda_i^{m-2}\lambda_p^m|D_{\bar i}g_{\bar ii}|^2\\
&\geq P_m\lambda_i^{m-2}\sum_{p\neq i}\sigma_k^{p\bar p}|D_ig_{\bar pp}|^2\geq 0.
\end{aligned}$$

This completes the proof of Lemma 2. Q.E.D.

We observed in (3.12) that $A_i\geq 0$. Lemma 2 implies that for any $i\neq 1$,

$$A_i+B_i+C_i+D_i-E_i\geq 0.$$

Thus we have shown that for $i\neq 1$, the third order terms in the main inequality (3.17) are indeed nonnegative. The only remaining case is when $i=1$. By adapting once again the techniques from [10], we obtain the following lemma.

Lemma 3 *Let $1<k\leq n$. Suppose there exists $0<\delta\leq 1$ such that $\lambda_\mu\geq\delta\lambda_1$ for some $\mu\in\{1,2,\ldots,k-1\}$. There exists a small $\delta'>0$ such that if $\lambda_{\mu+1}\leq\delta'\lambda_1$, then*

$$A_1+B_1+C_1+D_1-E_1\geq 0.$$

Proof. By Lemma 2, we have

$$\begin{aligned}
&P_m^2(A_1+B_1+C_1+D_1-E_1)\\
\geq\ &P_m^2A_1+P_m\lambda_1^{m-2}\sum_{p\neq 1}\sigma_k^{p\bar p}|D_1g_{\bar pp}|^2-\lambda_1^m\sigma_k^{1\bar 1}\lambda_1^{m-2}|D_1g_{\bar 11}|^2.
\end{aligned}\tag{3.32}$$

The key insight in [10], used also in [14], is to extract a good term involving $|D_1g_{\bar 11}|^2$ from A_1. By the inequality in Lemma 1 (with $\theta=\frac{1}{2}$), we have for $\mu<k$

$$\begin{aligned}
P_m^2A_1 &\geq \frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\Big\{(1+\frac{\alpha}{2})\Big|\sum_p\sigma_\mu^{p\bar p}D_1g_{\bar pp}\Big|^2-\sigma_\mu\sigma_\mu^{p\bar p,q\bar q}D_1g_{\bar pp}D_{\bar 1}g_{\bar qq}\Big\}\\
&= \frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\Big\{\sum_p(1+\frac{\alpha}{2})|\sigma_\mu^{p\bar p}D_1g_{\bar pp}|^2+\sum_{p\neq q}\frac{\alpha}{2}\sigma_\mu^{p\bar p}D_1g_{\bar pp}\sigma_\mu^{q\bar q}D_{\bar 1}g_{\bar qq}\\
&\quad +\sum_{p\neq q}(\sigma_\mu^{p\bar p}\sigma_\mu^{q\bar q}-\sigma_\mu\sigma_\mu^{p\bar p,q\bar q})D_1g_{\bar pp}D_{\bar 1}g_{\bar qq}\Big\}\\
&\geq \frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\Big\{\sum_p|\sigma_\mu^{p\bar p}D_1g_{\bar pp}|^2-\sum_{p\neq q}|F^{pq}D_1g_{\bar pp}D_{\bar 1}g_{\bar qq}|\Big\},
\end{aligned}\tag{3.33}$$

where we defined $F^{pq} = \sigma_\mu^{p\bar{p}}\sigma_\mu^{q\bar{q}} - \sigma_\mu \sigma_\mu^{p\bar{p},q\bar{q}}$. Notice if $\mu = 1$, then $F^{pq} = 1$. If $\mu \geq 2$, then the Newton-MacLaurin inequality implies

$$F^{pq} = \sigma_{\mu-1}^2(\lambda|pq) - \sigma_\mu(\lambda|pq)\sigma_{\mu-2}(\lambda|pq) \geq 0. \tag{3.34}$$

We split the sum involving F^{pq} in the following way:

$$\sum_{p\neq q} |F^{pq} D_1 g_{\bar{p}p} D_{\bar{1}} g_{\bar{q}q}| = \sum_{p\neq q; p,q\leq\mu} F^{pq}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| + \sum_{(p,q)\in J} F^{pq}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| \tag{3.35}$$

where J is the set of indices where at least one of $p \neq q$ is strictly greater than μ. The summation of terms in J can be estimated by

$$\begin{aligned} -\sum_{(p,q)\in J} F^{pq}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| &\geq -\sum_{(p,q)\in J} \sigma_\mu^{p\bar{p}}\sigma_\mu^{q\bar{q}}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| \\ &\geq -\epsilon \sum_{p\leq\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 - C\sum_{p>\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \end{aligned} \tag{3.36}$$

If $\mu = 1$, the first term on the right hand side of (3.35) vanishes and this estimate applies to all terms on the right hand side of (3.35).

If $\mu \geq 2$, we have for $p, q \leq \mu$,

$$\sigma_{\mu-1}(\lambda|pq) \leq C\frac{\lambda_1 \cdots \lambda_{\mu+1}}{\lambda_p \lambda_q} \leq C\frac{\sigma_\mu^{p\bar{p}}\lambda_{\mu+1}}{\lambda_q}. \tag{3.37}$$

Using (3.34) and (3.37), for δ' small enough we can control

$$\begin{aligned} -\sum_{p\neq q;p,q\leq\mu} F^{pq}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| &\geq -\sum_{p\neq q;p,q\leq\mu} \sigma_{\mu-1}^2(\lambda|pq)|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| \\ \geq -C\lambda_{\mu+1}^2 \sum_{p\neq q;p,q\leq\mu} \frac{\sigma_\mu^{p\bar{p}}}{\lambda_p}|D_1 g_{\bar{p}p}|\frac{\sigma_\mu^{q\bar{q}}}{\lambda_q}|D_{\bar{1}} g_{\bar{q}q}| &\geq -C\sum_{p\leq\mu}\frac{\lambda_{\mu+1}^2}{\lambda_p^2}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 \\ \geq -C\sum_{p\leq\mu}\frac{\delta'^2}{\delta^2}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 &\geq -\epsilon\sum_{p\leq\mu}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \end{aligned} \tag{3.38}$$

Combining all cases, we have

$$-\sum_{p\neq q}|F^{pq} D_1 g_{\bar{p}p} D_{\bar{1}} g_{\bar{q}q}| \geq -2\epsilon\sum_{p\leq\mu}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 - C\sum_{p>\mu}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \tag{3.39}$$

Using this inequality in (3.33) yields

$$\begin{aligned} P_m^2 A_1 &\geq \frac{P_m \lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\Big\{(1-2\epsilon)\sum_{p\leq\mu}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 - C\sum_{p>\mu}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2|\Big\} \\ &\geq (1-2\epsilon)\frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}|\sigma_\mu^{1\bar{1}} D_1 g_{\bar{1}1}|^2 - C\frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\sum_{p>\mu}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \end{aligned} \tag{3.40}$$

We estimate

$$
\begin{aligned}
&(1-2\epsilon)\frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}|\sigma_\mu^{1\bar{1}}D_1 g_{\bar{1}1}|^2 = (1-2\epsilon)\frac{P_m\lambda_1^{m-2}\sigma_k}{\lambda_1}\Big(\frac{\lambda_1\sigma_\mu^{1\bar{1}}}{\sigma_\mu}\Big)^2|D_1 g_{\bar{1}1}|^2 \\
\geq\ &(1-2\epsilon)P_m\lambda_1^{m-2}\frac{\sigma_k}{\lambda_1}\Big(1-C\frac{\lambda_{\mu+1}}{\lambda_1}\Big)^2|D_1 g_{\bar{1}1}|^2 \geq (1-2\epsilon)(1-C\delta')^2 P_m\lambda_1^{m-2}\sigma_k^{1\bar{1}}|D_1 g_{\bar{1}1}|^2 \\
\geq\ &(1-2\epsilon)(1-C\delta')^2(1+\delta^m)\lambda_1^{2m-2}\sigma_k^{1\bar{1}}|D_1 g_{\bar{1}1}|^2. \qquad (3.41)
\end{aligned}
$$

For δ' and ϵ small enough, we obtain

$$
P_m^2 A_1 \geq \lambda_1^m\sigma_k^{1\bar{1}}\lambda_1^{m-2}|D_1 g_{\bar{1}1}|^2 - C\frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\sum_{p>\mu}|\sigma_\mu^{p\bar{p}}D_1 g_{\bar{p}p}|^2. \qquad (3.42)
$$

We see that the $|D_1 g_{\bar{1}1}|^2$ term cancels from inequality (3.32) and we are left with

$$
P_m^2(A_1+B_1+C_1+D_1-E_1) \geq P_m\lambda_1^{m-2}\sum_{p>\mu}\Big\{\sigma_k^{p\bar{p}} - C\frac{\lambda_1\sigma_k(\sigma_\mu^{p\bar{p}})^2}{\sigma_\mu^2}\Big\}|D_1 g_{\bar{p}p}|^2. \qquad (3.43)
$$

For δ' small enough, the above expression is nonnegative. Indeed, for any $p>\mu$, we have

$$
(\lambda_1\sigma_\mu^{p\bar{p}})^2 \leq \frac{1}{\delta^2}(\lambda_\mu\sigma_\mu^{p\bar{p}})^2 \leq C\frac{(\sigma_\mu)^2}{\delta^2}, \qquad (3.44)
$$

Therefore

$$
C\frac{\lambda_1\sigma_k(\sigma_\mu^{p\bar{p}})^2}{\sigma_\mu^2} \leq \frac{C}{\delta^2}\frac{\sigma_k}{\lambda_1}. \qquad (3.45)
$$

On the other hand, we notice that, if $p>k$, then $\sigma_k^{p\bar{p}} \geq \lambda_1\cdots\lambda_{k-1} \geq c_n\frac{\sigma_k}{\lambda_k} \geq \frac{c_n}{\delta'}\frac{\sigma_k}{\lambda_1}$. If $\mu<p\leq k$, then $\sigma_k^{p\bar{p}} \geq \frac{\lambda_1\cdots\lambda_k}{\lambda_p} \geq c_n\frac{\sigma_k}{\lambda_p} \geq \frac{c_n}{\delta'}\frac{\sigma_k}{\lambda_1}$. It follows that for δ' small enough we have

$$
\sigma_k^{p\bar{p}} \geq C\frac{\lambda_1\sigma_k(\sigma_\mu^{p\bar{p}})^2}{\sigma_\mu^2}. \qquad (3.46)
$$

This completes the proof of Lemma 3. Q.E.D.

3.2 Completing the Proof

With Lemma 2 and Lemma 3 at our disposal, we claim that we may assume in inequality (3.17) that

$$
A_i+B_i+C_i+D_i-E_i \geq 0, \quad \forall i=1,\cdots,n. \qquad (3.47)
$$

Indeed, first set $\delta_1=1$. If $\lambda_2\leq\delta_2\lambda_1$ for $\delta_2>0$ small enough, then by Lemma 3 we see that (3.47) holds. Otherwise, $\lambda_2\geq\delta_2\lambda_1$. If $\lambda_3\leq\delta_3\lambda_1$ for $\delta_3>0$ small enough, then by Lemma 3 we see that (3.47) holds. Otherwise, $\lambda_3\geq\delta_3\lambda_1$. Proceeding iteratively, we may

arrive at $\lambda_k \geq \delta_k \lambda_1$. But in this case, the C^2 estimate follows directly from the equation as

$$C \geq \sigma_k \geq \lambda_1 \cdots \lambda_k \geq (\delta_k)^{k-1}\lambda_1. \tag{3.48}$$

Therefore we may assume (3.47), and inequality (3.17) becomes

$$\begin{aligned} 0 \ \ \geq \ \ & \frac{-C(K)}{\lambda_1}\Big\{1+|DDu|^2+|D\bar{D}u|^2\Big\}+(N-C\tau m N^2)(|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega}) \\ & +(M\varepsilon - C\tau m M^2 - CN - C)\mathcal{F} - CM. \end{aligned} \tag{3.49}$$

Since for fixed i, $\sigma_k^{i\bar{i}} \geq \sigma_k^{1\bar{1}} \geq \frac{k}{n}\frac{\sigma_k}{\lambda_1} \geq \frac{1}{C\lambda_1}$, we can estimate

$$|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega} \geq \frac{1}{C\lambda_1}(|DDu|^2+|D\bar{D}u|^2) \geq \frac{1}{C\lambda_1}|DDu|^2+\frac{\lambda_1}{C}. \tag{3.50}$$

This leads to

$$\begin{aligned} 0 \ \ \geq \ \ & \Big\{\frac{N}{C}-C\tau m N^2-C(K)\Big\}\lambda_1+\frac{1}{\lambda_1}\Big\{\frac{N}{C}-C\tau m N^2-C(K)\Big\}\Big\{1+|DDu|^2\Big\} \\ & +(M\varepsilon - C\tau m M^2 - CN - C)\mathcal{F} - CM. \end{aligned}$$

By choosing τ small, for example, $\tau = \frac{1}{NM}$, we have

$$\begin{aligned} 0 \ \ \geq \ \ & \Big\{\frac{N}{C}-\frac{Cm}{M}N-C(K)\Big\}\lambda_1+\frac{1}{\lambda_1}\Big\{\frac{N}{C}-\frac{Cm}{M}N-C(K)\Big\}\Big\{1+|DDu|^2\Big\} \\ & +(M\varepsilon - \frac{Cm}{N}M - CN - C)\mathcal{F} - CM. \end{aligned}$$

Taking N and M large enough, we can make the coefficients of the first three terms to be positive. For example, if we let $M = N^2$ for N large, then $\frac{N}{C}-\frac{Cm}{M}N-C(K) = \frac{N}{C}-\frac{Cm}{N}-C(K) > 0$ and $M\varepsilon-\frac{Cm}{N}M-CN-C = N^2\varepsilon-CmN-CN-C > 0$. Thus, an upper bound of λ_1 follows. Q.E.D.

Remark 1 *In the above estimate, we assume that $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_n$. Indeed, our estimate still works with $\lambda \in \Gamma_{k+1}$. It was observed in [14] (Lemma 7) that if $\lambda \in \Gamma_{k+1}$, then $\lambda_1 \geq \cdots \geq \lambda_n > -K_0$ for some positive constant K_0. Thus, we can replace λ by $\tilde{\lambda} = \lambda + K_0 I$ in our test function G in (3.2).*

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